Mathematical Physics
Individual Overall Contest

June 10, 2023

You can choose one of the following two questions.

Question 1. Consider a d-dimensional Ising model on a square lattice, which is periodic in all directions. The Hamiltonian is given by

$$H = -\sum_{a,b} J_{ab}\sigma_a\sigma_b - \sum_a B_a\sigma_a$$

where B_a is an external magnetic field, $\sigma_a = \pm 1$, and the spin-spin coupling $J_{ab} \equiv J(|\vec{a} - \vec{b}|)$ fall off with distance.

(1) Compute the classical partition function when $J_{ab} = 0$ first, then show that for non-vanishing J, this can be written as

$$Z = \mathcal{N}\int \mathcal{D}\psi\, e^{-S[\psi]}, \qquad S[\psi] = \sum_{a,b}\psi_a K_{ab}\psi_b - \sum_a \beta\psi_a B_a - \ln\cosh(2\sum_b K_{ab}\psi_b)$$

where $K = (\beta J)^{-1}$

(2) Show that at low temperature

$$S = \sum_{\mathbf{k}}[\phi_{\mathbf{k}}(c_0 + c_1\mathbf{k}\cdot\mathbf{k})\psi_{-\mathbf{k}} + c_2\psi_{\mathbf{k}}B_{-\mathbf{k}}] + c_3\sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4}\psi_{\mathbf{k}_1}\psi_{\mathbf{k}_2}\psi_{\mathbf{k}_3}\psi_{\mathbf{k}_4}\delta_{\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4} + \cdots$$

where

$$\psi_a = \frac{1}{\sqrt{N}}\sum_{\mathbf{k}} e^{-i\mathbf{k}\cdot a}\psi_{\mathbf{k}} \qquad K_{ab} = \frac{1}{N}\sum_{\mathbf{k}} e^{-i\mathbf{k}\cdot(a-b)}K_{\mathbf{k}}$$

then find coefficents c_i and explain when you can neglect the ellipsis in above expression.

(3) For vanishing external magnetic field, upon Fourier transforming and rescaling, the action become a ϕ^4 theory

$$Z = Z_0\int \mathcal{D}\phi\, e^{-S[\phi]} \qquad S[\phi] = \int d^dx\left(\frac{1}{2}(\partial\phi)^2 + \frac{m}{2}\phi^2 + \lambda\phi^4\right)$$

(4) At what temperature does the Ising model have a transition? (HINT: the Ising model has a transition when m changes sign)

Question 2. Consider a free massless Dirac fermion in two-dimensions with Lagrangian

$$\mathcal{L} = i\bar{\psi}\partial\!\!\!/\psi$$

The vector current $J_\mu \equiv \bar{\psi}\gamma_\mu\psi$ is conserved $\partial^\mu J_\mu = 0$. By the Poincaré lemma this means that we can introduce a scalar operator (field) $\phi(x)$ such that

$$J^\mu(x) = \epsilon^{\mu\nu}\partial_\nu\phi(x).$$

(1) Answer to ***either one*** of the following two questions (they are equivalent, just stated in two different languages so that you can use the one which you feel more convenient):

   (a) using the symmetries of the theory, write down the dynamical equations of the 2d Dirac theory as a field equation for the scalar field $\phi(x)$;

   (b) compute the two-point function of $\phi(x)$ from the identity

   $$\langle J^{\mu_1}(x_1)\, J^{\mu_2}(x_2)\rangle = \epsilon^{\mu_1\nu_1}\epsilon^{\mu_2,\nu_2}\partial_{x_1^{\nu_1}}\partial_{x_2^{\nu_2}}\langle\phi(x_1)\,\phi(x_2)\rangle$$

   [HINT: working in momentum space is more convenient]

(2) can you write a local action for the field $\phi(x)$ whose variation yields the field equations obtained in (1)?

(3) discuss briefly what your answer to (1) teaches us regarding the relation of bosons to fermions in two-dimensions

(4) is your result (and discussion) consistent with the *Spin & Statistics* theorem of QFT?

(5) Now couple your massless 2d Dirac fermion to an Abelian gauge field A_μ

$$\mathcal{L} = -\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\gamma^\mu(\partial_\mu - iA_\mu)\psi.$$

What is the particle spectrum of the resulting theory ($\equiv$ massless 2d QED)? **To avoid all computation, the numerical value of the masses of the various particles is <u>NOT</u> required. Just say for each particle whether its mass is zero or non-zero.**